“JACOBIANS ”

Guided By

- **Prof. V. S. Bawane**

Mahatma Fule Arts, Comm. & Sitaramji Chaudhari Science Mahavidyalaya, Warud.

Department of Mathematics

ADVANCED CALCULUS

JACOBIANS

Prof. V. S. Bawane
Head

Contents

Introduction

Jacobian

- The **Jacobian determinant**, or just a **Jacobian**, is the determinant of the matrix of partial derivatives of a system of equations.

LITERATURE SURVEY

- The inventor of Jacobian's is
  Carl Gustav Jacob Jacobi.
- **Born**: 10 December 1804 Potsdam,
  Kingdom of Prussia
- Died: 18 February 1851(aged 46)
  Berlin, Kingdom of Prussia
- Known for: Jocobi's elliptic function
  - Jacobian
  - Jacobi symbol
  - Jacobi identity
  - Jacobi operator.

Carl Gustav Jacob Jacobi

THEORY

Let, u(x,y) & v(x,y) be differential function of independent variable x and y defined by u=f(x,y) & v=g(x,y).

Then, a determinant,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called as Jacobian determinant or Jacobian transformation or simply Jacobian of,

u=f(x,y) & v=g(x,y).

It is denoted by,

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

The Jacobian of inverse mapping x, y w.r.to u, v is given by

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$And \quad J \cdot J' = 1$$

Similarly,

The Jacobian of u=f(x,y,z), v=g(x,y,z), w=h(x,y,z) with respect to x,y,z is given by,

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Solved Problems

$Ex.1)\ If\ \ x = r\cos\theta, y = r\sin\theta\ \ then\ \ show\ \ that\ \ \frac{\partial(x,y)}{\partial(r,\theta)} = r$

$Sol^n \Rightarrow \quad Let \quad x = r\cos\theta, y = r\sin\theta$

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$x = r\cos\theta, y = r\sin\theta \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Ex.2) *Find the Jacobian of the mapping*

$$u = 2x - y, v = x + 4y$$

Hence find the Jacobian of inverse mapping.

Solution :

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9$$

The Jacobian of inverse mapping is

$$J' = \frac{1}{J} = \frac{1}{9}$$

Example.3 $x+y+z=u,\quad y+z=uv,\quad z=uvw$

Prove that: $$\frac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$$

⇒

$$x+y+z=u \qquad y+z=uv \qquad z=uvw$$

$$x=u-(y+z) \qquad y=uv-z \qquad z=uvw$$

$$x=u-uv \qquad y=uv-uvw \qquad z=uvw$$

$$x=u(1-v) \qquad y=uv(1-w) \qquad z=uvw$$

$$\therefore \frac{\partial x}{\partial u}=1-v \qquad \therefore \frac{\partial y}{\partial u}=v(1-w) \qquad \therefore \frac{\partial z}{\partial u}=vw$$

$$\therefore \frac{\partial x}{\partial v}=-u \qquad \therefore \frac{\partial y}{\partial v}=u(1-w) \qquad \therefore \frac{\partial z}{\partial v}=uw$$

$$\therefore \frac{\partial x}{\partial w}=0 \qquad \therefore \frac{\partial y}{\partial w}=-uv \qquad \therefore \frac{\partial z}{\partial w}=uv$$

$$
= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}
$$

$$
= \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix}
$$

$$J = 1 - v[uv \cdot u(1-w) + uv \cdot uw]$$
$$+ u[uv \cdot v(1-w) - vwv(1-w)] + 0$$

$$= (1-v)[u^2 v - u^2 vw + u^2 vw]$$
$$+ u[uv^2 - uv^2 w + uv^2 w]$$

$$= (1-v)(u^2 v) + u^2 v^2$$

$$= u^2 v - u^2 v^2 + u^2 v^2$$

$$= u^2 v$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$$

Theorem (Chain Rule)

If x,y are differentiable functions of u, v and u, v are differentiable functions of r, s then,

$$\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(r,s)}=\frac{\partial(x,y)}{\partial(r,s)}$$

Proof : Let $x = f_1(u,v), y = f_2(u,v)$ and

$u = \phi_1(r,s), v = \phi_2(r,s)$ be differencial functions.

Then x, y are diff. functions of r, s.

Differentiating x, y partially with respect to r, s we get,

$$\frac{\partial x}{\partial r} = \frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial r},$$

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial s}$$

$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial r},$$

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial s} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial s}$$

$$\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial r}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial r} & \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial s}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial s} \\ \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial r}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial r} & \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial s}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial s} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(r,s)}$$

APPLICATIONS:

- The Jacobian determinant is used to transform the multiple integrals by changing the variables to **polar, spherical polar, cylindrical** and curvilinear co-ordinates etc.

For Ex.

$$If\ x = r\cos\theta, y = r\sin\theta\ then$$

$$dxdy = |J| drd\theta$$

$$\iint f(x,y)dxdy = \iint f(r,\theta)|J|drd\theta = \iint f(r,\theta)rdrd\theta$$

$$where\ J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Transformation of triple integral to spherical polar co-ordinates.

$$Let\ the\ spherical\ polar\ co-ordinates\ are$$

$$x = r\cos\theta\sin\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\phi$$

$$then,$$

$$\iiint_v f(x,y,z)dxdydz = \iiint_v f(r,\theta,\phi)\left|J\right|drd\theta d\phi$$

$$and$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = -r^2\sin\phi drd\theta d\phi$$

The formula for change of variables in double integral from x,y to u,v is,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f\,[x(u,v), y(u,v)] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

That is the integrant is expressed in terms of u and v $dx\,dy$ is replaced by du dv times the absolute value of Jacobian,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

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Thank You……..!